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An Abstract Version of a Limit Theorem of Szegö

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The theory of factorization with respect to chains of orthogonal projections is used to deduce an abstract version of a well-known limit theorem of Szegö. The theory is then specialized to matrix Fredholm operators and finally to matrix Wiener–Hopf operators. The well-known continuous analogues of Kac and Achiezer emerge as special cases.

Contents.

1. INTRODUCTION

Let K be a bounded linear operator on a separable Hilbert space H and let $\mathbb{P}$ be a maximal chain of orthogonal projectors on H, as defined on p. 14 of Gohberg and Krein [10]. In this paper we shall study the growth of

$$\det[I - PKP]$$

as P increases along the chain $\mathbb{P}$ towards the identity. The first main result: Theorem 2.1, is an abstract version of a limit theorem which originated in the fundamental investigations of Szegö [16] on the growth of the determinant of a truncated Toeplitz matrix as the block size of the truncation tends to infinity.

In Section 3 the corresponding limit theorem for Fredholm integral operators acting on $L_n^2[0, \infty)$ is reexpressed in terms of resolvent kernels. Formulas (3.1) and (3.2) appear to be new. In Section 4 the results are further specialized to Wiener–Hopf operators, i.e., to the case in which the kernel of the integral operator is a translation kernel, and a number of related identities are noted. In particular the continuous analogues of Szegö's formula due to Kac [14] and Achiezer [1] and also Mikaelian's generalization of the latter [15] emerge as special cases. For additional discussion and generalizations see Dym [4–7].

We do not treat the Szegö formula on the circle. The best recent work on

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that subject has been done by Widom [17–19]; see also Basor and Helton [2] for an elegant argument and new results for discontinuous symbols, and Hirschman [13] for a survey and extensive bibliography of the earlier (up to about 1970) work on the subject.

We shall use the following notation: $\mathbb{R}$ and $\mathbb{C}$ will denote the real and complex numbers, respectively; $A^\times$ stands for the conjugate transpose of the matrix A and $|A|$ stands for its maximum s-number. $L^p_{n\times k}(J)$ for J any subinterval of $\mathbb{R}$ stands for the set of $n\times k$ matrix valued functions f with $|f|$ in the usual scalar L^p space over J with respect to Lebesgue measure; L^p_n is short for $L^p_{n\times 1}$. $\hat{f}(\lambda)=\int_{-\infty}^{\infty} f(x)\,e^{i\lambda x}\,dx$ and $f^\vee(s)=(1/2\pi)\int_{-\infty}^{\infty} f(\lambda)\,e^{-i\lambda s}\,ds$ denote the Fourier transform pair for suitably restricted f and e denotes the $n\times n$ identity matrix. Finally, ▮ marks the end of a proof.

2. The Main Theorem

In this section we shall establish the main limit theorem alluded to in the introduction, but pause first to recall some definitions and preliminary lemmas.

The operator $I-K$ will be said to admit an invertible left [resp. right] factorization with respect to the chain $\mathbb{P}$ if it can be expressed in the form

$$I-K=(I+X_+)(I+X_-) \qquad [\text{resp. } I-K=(I+X_-)(I+X_+)], \tag{2.1}$$

where $I+X_\pm$ are bounded invertible operators,

$$(I+X_\pm)^{-1}=I+Y_\pm \tag{2.2}$$

and, for every projection $P\in\mathbb{P}$,

$$PX_-=PX_-P, \qquad X_+P=PX_+P \tag{2.3}$$

and (2.3) holds with $Y_\pm$ in place of $X_\pm$.

We remark that we are *not* assuming at this point that the factors $X_\pm$ are Volterra operators and thus the factorization is not necessarily unique; see Chapter 4 of Gohberg and Krein [10] for additional discussion.

Before proceeding to the main theorem it is convenient to establish a number of lemmas.

Lemma 2.1. *If $I-K$ is invertible and PKQ and QKP are Hilbert–Schmidt, where $Q=I-P$ and $P\in\mathbb{P}$, then*

$$P-P(I-K)P(I-K)^{-1}P$$

is trace class.

Proof. Since the given operator is equal to

$$P(I-K)\,Q(I-K)^{-1}P = -PKQ(I-K)^{-1}P$$

and the product of two Hilbert–Schmidt operators is trace class it suffices to show that $Q(I-K)^{-1}P$ is Hilbert–Schmidt. But

$$\begin{aligned} Q(I-K)^{-1}P &= Q[(I-K)^{-1}-I]P \\ &= Q[K(I-K)^{-1}P] \\ &= QKP(I-K)^{-1}P + QKQ(I-K)^{-1}P \\ &= QKP(I-K)^{-1}P + KQ(I-K)^{-1}P - PKQ(I-K)^{-1}P, \end{aligned}$$

which in turn implies that

$$Q(I-K)^{-1}P = (I-K)^{-1}[QKP(I-K)^{-1}P - PKQ(I-K)^{-1}P]$$

which is clearly Hilbert–Schmidt, since the class of Hilbert–Schmidt operators is a two-sided ideal in the algebra of bounded linear operators on H. ∎

LEMMA 2.2. *If $I-K$ admits an invertible left factorization* (2.1) *and if PKQ and QKP are Hilbert–Schmidt for some $P \in \mathbb{P}$ and $Q = I-P$, then (for the very same P) the operators PX_+Q, QX_-P, PY_+Q and QY_-P are also Hilbert–Schmidt.*

Proof. It follows from (2.1) that

$$(I-K)(I+Y_-) = I + X_+$$

and hence that

$$-K + Y_- - KY_- = X_+.$$

Therefore, since

$$Y_-Q = QY_-Q$$

it follows that

$$-PKQ - PKQY_-Q = PX_+Q.$$

This proves the first assertion. The second assertion follows from the formula

$$(I+Y_+)(I-K) = I + X_-$$

in much the same way.

Next, the identity

$$(I+X_+)(I+Y_+)=I$$

implies that

$$X_+ + Y_+ + X_+Y_+ = 0$$

and hence that

$$PX_+Q + PY_+Q + PX_+Y_+Q = PX_+Q + PY_+Q + PX_+PY_+Q + PX_+QY_+Q = 0.$$

Therefore,

$$PX_+Q(I+QY_+Q) = -(I+PX_+P)PY_+Q$$

and so, as

$$(I+PY_+P)(I+PX_+P) = I,$$

it follows that

$$PY_+Q = -(I+PY_+P)(PX_+Q)(I+QY_+Q)$$

is Hilbert–Schmidt, since PX_+Q is. The proof for QY_-P is similar. ∎

LEMMA 2.3. *If $I-K$ admits an invertible left factorization* (2.1) *with respect to the chain* $\mathbb{P}$, *if PKQ and QKP are Hilbert–Schmidt and PKP is trace class for some $P \in \mathbb{P}$ and $Q = I-P$, then (for the very same P)*

$$PX_+P + PX_-P + PX_+PX_-P$$

and

$$PY_-P + PY_+P + PY_-PY_+P$$

are trace class and

$$\det[(I+PX_+P)(I+PX_-P)] \neq 0.$$

Proof. It follows from (2.1) that

$$\begin{aligned} A &= PX_+P + PX_-P + PX_+PX_-P \\ &= PX_+P + PX_-P + PX_+X_-P - PX_+QX_-P \\ &= -PKP - PX_+QX_-P \end{aligned}$$

which is trace class thanks to the assumptions, Lemma 2.2 and the fact that the product of two Hilbert–Schmidt operators is trace class. Thus $\det(I+A)$

is well defined. Moreover, since $I+A$ is invertible, and the set of trace class operators is a two-sided ideal in the algebra of bounded operators on H,

$$B=(I+A)^{-1}-I=(I+PY_-P)(I+PY_+P)-I$$

is also of trace class and, by rule 7 on p. 162 of [9],

$$1=\det[(I+A)(I+B)]=\det(I+A)\det(I+B).$$

Hence $\det(I+A)\neq 0$. ∎

THEOREM 2.1. *If $I-K$ admits an invertible left factorization* (2.1) *with respect to the chain $\mathbb{P}$, if PKQ and QKP are Hilbert-Schmidt and PKP is trace class for every $P\in\mathbb{P}$ except $P=I$ and if*

$$Z=(I+Y_-)(I-K)(I+Y_+)-I$$

is trace class, then

$$\lim_{P\to I}\frac{\det(I-PKP)}{\det[(I+PX_+P)(I+PX_-P)]}=\det(I+Z).$$

Proof. By rules 6 and 7 on page 162 of Gohberg and Krein [9]

$$\begin{aligned}\det(I-PKP)&=\det[(I+PY_-P)(I-PKP)(I+PX_-P)]\\&=\det[(I+PY_-P)(I-PKP)(I+PY_+P)(I+PX_+P)(I+PX_-P)]\\&=\det[(I+PZP)(I+PX_+P)(I+PX_-P)]\\&=\det(I+PZP)\det[(I+PX_+P)(I+PX_-P)]\end{aligned}$$

for every $P\in\mathbb{P}$ except $P=I$, thanks to the assumptions and Lemma 2.3 which insure that the operators in each of the determinants on the last line differs from the identity by a trace class operator. The final statement now follows upon dividing through by the last determinant, which is legitimate in view of Lemma 2.3, and letting $P\to I$. Corollary 1.1 on p. 160 of Gohberg and Krein [9] guarantees that

$$\det(I+PZP)\to\det(I+Z)$$

since $PZP\to Z$ in trace class as $P\to I$. ∎

We remark that the condition on Z in the last theorem is met, for example, if K is of Hilbert–Schmidt class and if $I-QKQ$ is invertible for every $P\in\mathbb{P}$, where $Q=I-P$. In this case, as follows from Theorems 10.1 of Chapter I and 6.2 of Chapter IV of Gohberg and Krein [10], $I-K$ admits an invertible left factorization of the form (2.1) with $X_{\pm}$ Hilbert–Schmidt. But this in turn implies that

$$-K - X_{+} - X_{-} = X_{+}X_{-}$$

is trace class. At the same time the identity

$$(I + X_{\pm})(I + Y_{\pm}) = I$$

implies that $Y_{\pm}$ is of Hilbert–Schmidt class and

$$X_{\pm} + Y_{\pm} = -X_{\pm}Y_{\pm}$$

is of trace class. Thus $-K + Y_{-} + Y_{+}$ is trace class as is Z.

We further remark that if, in the preceding theorem, the factors $PX_{\pm}P$ are Hilbert–Schmidt operators of the Volterra type (i.e., with spectral radius equal to zero), then

$$PX_{+}P + PX_{-}P \qquad \text{and} \qquad PY_{-}P + PY_{-}P$$

are trace class and

$$\begin{aligned}\det\{(I + PX_{+}P)(I + PX_{-}P)\} &= \exp\{\text{trace}\,[PX_{+}P + PX_{-}P]\}\\ &= \exp\{-\text{trace}\,[PY_{+}P + PY_{-}P]\}.\end{aligned}$$

This can be verified with the help of the theory of regularized determinants much as in the proof of Theorem 3.1.

3. The Fredholm Case

In this section we shall study the limit of $\det(I - P_T K P_T)$ as $T \uparrow \infty$ in the special case in which K is a bounded integral operator with suitably smooth $n \times n$ matrix valued kernel $K(t, s)$:

$$Kf(t) = \int_0^\infty K(t, s)\, f(s)\, ds \qquad (t \geqslant 0),$$

which acts on the Hilbert space $L_n^2[0, \infty)$ of complex $\mathbb{C}^n$ valued functions on the positive half-line with respect to the standard inner product and

$$\begin{aligned} P_T f(s) &= f(s) \qquad &&\text{if} \quad 0 \leqslant s \leqslant T\\ &= 0 &&\text{otherwise,}\end{aligned}$$

for $0 \leqslant T < \infty$. We shall use the notation $I + \Gamma_a^b$ for the resolvent of $I - K$ restricted to $L_n^2[a, b]$ whenever it exists.

THEOREM 3.1. *If K is a bounded integral operator on $L_n^2[0, \infty)$ with continuous kernel $K(t, s)$ and if $P_T K P_T$ is trace class and $I - P_t K P_t$ is invertible for all $0 \leqslant t \leqslant T$, then*

$$\det(I - P_t K P_t) = \exp\left\{ -\operatorname{trace} \int_0^t \Gamma_0^s(s, s)\, ds \right\}$$

for all $0 \leqslant t \leqslant T$.

Proof. Under the given assumptions, see, e.g., pages 183–187 of Gohberg and Krein [10], $I - P_T K P_T$ admits a right factorization:

$$(I - P_T K P_T) = (I + U_-)(I + U_+),$$

where $U_\pm$ and $V_\pm = (I + U_\pm)^{-1} - I$ are Hilbert–Schmidt operators on $L_n^2[0, T]$ with triangular kernels. In particular

$$\begin{aligned} V_+(t, s) &= \Gamma_0^s(t, s) \qquad &&\text{for} \quad 0 \leqslant t < s \\ &= 0 &&\text{for} \quad 0 \leqslant s < t \end{aligned}$$

and

$$\begin{aligned} V_-(t, s) &= 0 \qquad &&\text{for} \quad 0 \leqslant t < s \\ &= \Gamma_0^t(t, s) &&\text{for} \quad 0 \leqslant s < t. \end{aligned}$$

Therefore,

$$I - P_t K P_t = (I + P_t U_- P_t)(I + P_t U_+ P_t)$$

and

$$\det(I - P_t K P_t) = \widetilde{\det}(I - P_t K P_t) \exp\{-\operatorname{trace}[P_t K P_t]\},$$

where $\widetilde{\det}$ denotes the regularized determinant which is well defined for operators which differ from the identity by a Hilbert–Schmidt operator; see, pages 166–168 of Gohberg and Krein [9]. Moreover, by rule 3 on page 169 of [9], the right hand side of the last equality is equal to

$$\widetilde{\det}(I + P_t U_- P_t)\, \widetilde{\det}(I + P_t U_+ P_t) \exp\{\operatorname{trace}[P_t U_- P_t + P_t U_+ P_t]\}$$

which in turn is equal to just the exponential term since $P_t U_\pm P_t$ are Volterra operators. But now, as

$$(I + P_t U_\pm P_t)(I + P_t V_\pm P_t) = I$$

and $P_t U_\pm P_t V_\pm P_t$ is both Volterra and trace class, it follows that

$$\operatorname{trace}(P_t U_\pm P_t V_\pm P_t) = 0,$$

(see, e.g., Theorem 8.4 on page 101 of [9]), that

$$P_t U_- P_t + P_t U_+ P_t \qquad \text{and} \qquad P_t V_- P_t + P_t V_+ P_t$$

are trace class and that

$$\operatorname{trace}(P_t U_- P_t + P_t U_+ P_t) = -\operatorname{trace}(P_t V_- P_t + P_t V_+ P_t).$$

Finally, this can be evaluated with the help of Corollary 10.2 on page 117 of Gohberg and Krein [9] and the explicit formulas for the kernels of $V_\pm$ given above:

$$\begin{aligned}
&\operatorname{trace}[P_t V_- P_t + P_t V_+ P_t] \\
&= \sum_{j=1}^{n} \lim_{h \downarrow 0} \frac{1}{4h^2} \int_0^t \int_0^t [2h - |x-y|]_+ [\Gamma_0^x(x,y) + \Gamma_0^y(x,y)]_{jj}\, dx\, dy \\
&= \sum_{j=1}^{n} \int_0^t [\Gamma_0^s(s,s)]_{jj}\, ds \\
&= \operatorname{trace} \int_0^t \Gamma_0^s(s,s)\, ds. \quad \blacksquare
\end{aligned}$$

THEOREM 3.2. *If K is a bounded integral operator on $L_n^2[0,\infty)$ with continuous kernel $K(t,s)$ and if $I-K$ admits an invertible left factorization*

$$I - K = (I + X_+)(I + X_-)$$

with factors $Y_\pm = (I + X_\pm)^{-1} - I$ which have continuous kernels $Y_\pm(t,s)$ on their supporting triangles, and if $P_T K Q_T$ and $Q_T K P_T$ are Hilbert–Schmidt and $P_T K P_T$ is trace class, then

$$\det[(I + P_T X_+ P_T)(I + P_T X_- P_T)] = \exp\left\{-\operatorname{trace} \int_0^T \Gamma_s^\infty(s,s)\, ds\right\}.$$

Proof. Since $P_T X_\pm P_T$ are Volterra operators of Hilbert–Schmidt class and hence, by Lemma 2.3, $P_T X_+ P_T + P_T X_- P_T$ is of trace class, it follows, much as in the proof of Theorem 3.1, that

$$\begin{aligned}
\det[(I + P_T X_+ P_T)(I + P_T X_- P_T)] &= \exp\{\operatorname{trace}[P_T X_+ P_T + P_T X_- P_T]\} \\
&= \exp\{-\operatorname{trace}[P_T Y_+ P_T + P_T Y_- P_T]\}.
\end{aligned}$$

The final formula now follows from Corollary 10.2 on page 117 of Gohberg and Krein [9], just as in the proof of Theorem 3.1, and the explicit identification of the kernels as

$$\begin{aligned} Y_+(t,s) &= \Gamma_t^\infty(t,s) && \text{for} \quad 0 \leqslant t < s \\ &= 0 && \text{for} \quad 0 \leqslant s < t, \\ Y_-(t,s) &= 0 && \text{for} \quad 0 \leqslant t < s \\ &= \Gamma_s^\infty(t,s) && \text{for} \quad 0 \leqslant s < t. \end{aligned}$$

We remark that the indicated resolvents in Theorem 3.2 exist because, as follows from the assumed factorization, $I - Q_t K Q_t$ is invertible for every $t \geqslant 0$. The actual identifications can be carried out much as on pages 183–186 of Gohberg and Krein [10]. ∎

THEOREM 3.3. *If K is a bounded integral operator on $L_n^2[0,\infty)$ with continuous kernel $K(t,s)$ such that the restriction of $I - K$ to $L_n^2(a,b)$ is an invertible map of that space into itself for every $b \geqslant a \geqslant 0$ and if K meets the hypotheses of Theorems* 3.1 *and* 3.2, *then*

$$\frac{det(I - P_T K P_T)}{\det[(I + P_T X_+ P_T)(I + P_T X_- P_T)]} = \exp\left\{\text{trace} \int_0^T \left[\int_T^\infty \Gamma_t^s(t,s)\, \Gamma_t^s(s,t)\, ds\right] dt\right\}.$$

If K meets the hypotheses of Theorem 2.1., *then the limit as $T \uparrow \infty$ exists and is finite.*

Proof. By Theorems 3.1 and 3.2 the ratio on the left hand side of the asserted identity is equal to

$$\exp\left\{-\text{trace} \int_0^T [\Gamma_0^t(t,t) - \Gamma_t^\infty(t,t)]\, dt\right\}.$$

Moreover, a variant of the Bellman–Krein identity (see, e.g., pages 186, 187 of Gohberg and Krein [10] for the latter) implies that

$$\frac{\partial}{\partial s} \Gamma_s^t(t,t) = -\Gamma_s^t(t,s)\, \Gamma_s^t(s,t)$$

for $s < t$ and hence that

$$\Gamma_0^t(t,t) = K(t,t) + \int_0^t \Gamma_s^t(t,s)\, \Gamma_s^t(s,t)\, ds,$$

whereas the Bellman–Krein identity itself implies that

$$\frac{\partial}{\partial u}\Gamma_t^u(t,t)=\Gamma_t^u(t,u)\,\Gamma_t^u(u,t)$$

for $u>t$ and hence that

$$\Gamma_t^\infty(t,t)=K(t,t)+\int_t^\infty \Gamma_t^u(t,u)\,\Gamma_t^u(u,t)\,du.$$

Thus

$$\begin{aligned}\int_0^T &|\Gamma_0^t(t,t)-\Gamma_t^\infty(t,t)|\,dt\\ &=\int_0^T\left\{\int_0^t \Gamma_s^t(t,s)\,\Gamma_s^t(s,t)\,ds-\int_t^\infty \Gamma_t^s(t,s)\,\Gamma_t^s(s,t)\,ds\right\}dt.\end{aligned}$$

But the trace of the first term on the right of the last equality is equal to

$$\begin{aligned}\operatorname{trace}\int_0^T\left\{\int_s^T \Gamma_s^t(t,s)\,\Gamma_s^t(s,t)\,dt\right\}ds&=\operatorname{trace}\int_0^T\left\{\int_t^T \Gamma_t^s(s,t)\,\Gamma_t^s(t,s)\,ds\right\}dt\\ &=\operatorname{trace}\int_0^T\left\{\int_t^T \Gamma_t^s(t,s)\,\Gamma_t^s(s,t)\,ds\right\}dt\end{aligned}$$

as follows upon first changing the order of integration, then interchanging the variables and finally making use of the fact that trace $AB=$ trace BA for $n\times n$ matrices A and B. The final formula drops out upon combining this with the trace of the second term in the preceding equality. The existence of the limit is guaranteed by Theorem 2.1. ∎

We remark that the limit in the statement of Theorem 3.3 can also be expressed as

$$\exp\left\{-\operatorname{trace}\int_0^\infty\left[\int_0^\infty \operatorname{sign}(t-s)\,\varphi(t,s)\,ds\right]dt\right\},$$

where

$$\begin{aligned}\varphi(t,s)&=\Gamma_s^t(t,s)\,\Gamma_s^t(s,t)\qquad &&\text{for}\quad t\geqslant s\geqslant 0\\ &=\Gamma_t^s(t,s)\,\Gamma_t^s(s,t)\qquad &&\text{for}\quad s\geqslant t\geqslant 0.\end{aligned}$$

This is immediate from the formulas presented in the proof of the theorem.

We remark that the conditions of Theorem 3.1 will be met if, in addition to being continuous, the $n\times n$ matrix kernel $K(t,s)$ is such that

(1) there exists a constant M_p, for $p=1, 2$, such that

$$\|K(t, \cdot\,)\|_p \leqslant M_p \qquad \text{and} \qquad \|K(\,\cdot\,, t)\|_p \leqslant M_p$$

independently of $t \geqslant 0$ and $M_1 < \frac{1}{2}$;

(2) $K(t, \cdot\,)$ and $K(\,\cdot\,, t)$ are continuous in $L^2_{n\times n}[0, \infty)$;

(3) $\iint_0^\infty \{|K(t, s)|^2\}\, dt\, ds < \infty$;

(4) for each $T \geqslant 0$ there exists a pair of constants c_T and α_T, $\frac{1}{2} < \alpha_T \leqslant 1$, such that

$$|K(t, s_2) - K(t, s_1)| \leqslant c_T |s_2 - s_1|^{\alpha_T}$$

for $0 \leqslant s_1, s_2 \leqslant T$.

The proof is based in part on the inequality

$$\left| \int_0^\infty K(t, s) f(s)\, ds \right| \leqslant \int_0^\infty |K(t,s)|\, |f(s)|\, ds$$

(in which, just as above, $|\;|$ stands for the maximum s-number of the indicated matrix valued function), which serves to convert matrix estimates to scalar estimates. In particular it follows readily from assumption (1) that K and K^* are bounded mappings of $L^2_n[0, \infty)$ into itself (in fact $L^p_n[0, \infty)$ into itself for every $1 \leqslant p \leqslant \infty$) with norm less than or equal to M_1. Hence if $M_1 < \frac{1}{2}$, then $\Gamma = (I-K)^{-1} - I$ has norm less than one and so the restrictions of both $I-K$ and $I+\Gamma$ to any subspace $L^2_n[a, b]$ are automatically invertible. Continuity of the appropriate resolvent kernels follows from (1) and (2) and the resolvent equations. Assumption (3) guarantees that K is Hilbert–Schmidt and hence also, by the discussion following Theorem 2.1, that Z is trace class. Assumption (4) guarantees that $P_T K P_T$ is trace class by a result due to Stinespring; see p. 119 of [9] for discussion and the references.

Assumptions (1) to (4) can be relaxed somewhat at the expense of more careful estimates and also if, as will be the case in the next section, the kernel has additional structure.

4. Translation Kernels

In this section we specialize the main formulas of the previous sections to the case in which the kernel of K is a continuous translation kernel, i.e., $K(t, s) = k(t-s)$, where k is a continuous $n \times n$ matrix valued function on $\mathbb{R}$.

LEMMA 4.1. *If K has a continuous translation kernel and if the restriction of $I-K$ to $L_n^2[a,b]$ is an invertible map of $L_n^2[a,b]$ onto itself, then $I-K$ defines an invertible map of $L_n^2[0,b-a]$ onto itself and the corresponding resolvent kernels are related by the formula*

$$\Gamma_a^b(a+s, a+t) = \Gamma_0^{b-a}(s,t)$$

for $0 \leqslant s,t \leqslant b-a$.

Proof. By an elementary change of variables the resolvent equation for translation kernels:

$$\Gamma_a^b(u,v) - k(u-v) - \int_a^b k(u-w)\,\Gamma_a^b(w,v)\,dw = 0,$$

for $a \leqslant u, v \leqslant b$, can be reexpressed in the form

$$\Gamma_a^b(a+s, a+t) - k(s-t) - \int_0^{b-a} k(s-\tau)\,\Gamma_a^b(a+\tau, a+t)\,d\tau = 0,$$

for $0 \leqslant s,t \leqslant b-a$. The identification follows by comparison with the resolvent equation for the interval $[0, b-a]$. ∎

COROLLARY 1. *If K has a continuous translation kernel and if $I-K$ is an invertible map of $L_n^2[0,\infty)$ onto itself, then the restriction of $I-K$ to $L_n^2[t,\infty)$ is an invertible map of that space onto itself for every $t \geqslant 0$ and*

$$\Gamma_t^\infty(t+s, t) = \Gamma_0^\infty(s, 0) = a(s), \qquad s \geqslant 0,$$

where $a(s)$ is the solution of the Wiener–Hopf equation

$$a(s) - \int_0^\infty k(s-u)\,a(u)\,du = k(s)$$

for $s \geqslant 0$.

LEMMA 4.2. *If $I-K$ is an invertible map of $L_n^2[0,\infty)$ onto itself and if $K(t,s) = k(t-s)$ and both k and $\hat{k}$ belong to $L_{n\times n}^1(\mathbb{R})$, then $K(t,s)$ is continuous, $I - Q_t K Q_t$ is invertible for every $t \geqslant 0$, where $Q_t = I - P_t$ and*

$$-\operatorname{trace} \Gamma_0^\infty(0,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \log\det[e - \hat{k}(\lambda)]\,d\lambda. \tag{4.1}$$

Proof. It is well known, see, e.g., Theorem 6.1 on page 206 of Gohberg and Feldman [11] that, for any p, $1 \leqslant p \leqslant \infty$, $I-K$ is an invertible map of

$L_n^p[0, \infty)$ onto itself if and only if $e - \hat{k}(\lambda)$ admits a representation of the form

$$e - \hat{k}(\lambda) = [e + \hat{b}(\lambda)]^{-1}[e + \hat{a}(\lambda)]^{-1} \qquad (\lambda \in \mathbb{R}), \tag{4.2}$$

where $b \in L^1_{n \times n}(-\infty, 0]$, $a \in L^1_{n \times n}[0, \infty)$ and $\det[e + \hat{a}(\lambda)]$ [resp. $\det[e + \hat{b}(\lambda)]$] has no roots in the closed upper [resp. lower] half-plane. Moreover, a and b are uniquely specified as the solutions of

$$a(t) - \int_0^\infty k(t-s)\, a(s)\, ds = k(t) \qquad \text{for} \quad t \geqslant 0$$

and

$$b(t) - \int_{-\infty}^0 b(s)\, k(t-s)\, ds = k(t) \qquad \text{for} \quad t \leqslant 0.$$

This in turn implies by easy estimates (since k and $\hat{k}$ belong to $L^1_{n \times n}$) that a and b are continuous on their respective supporting half-lines, that

$$a(s) = \Gamma_0^\infty(s, 0) \qquad \text{and} \qquad b(-s) = \Gamma_0^\infty(0, s)$$

for $s \geqslant 0$ and hence in particular that

$$a(0) \equiv a(0+) = \Gamma_0^\infty(0, 0) = b(0-) \equiv b(0).$$

Now, since $\det[e - \hat{k}(\lambda)] \neq 0$ for $\lambda \in \mathbb{R}$ and $\hat{k} \in L^1_{n \times n}$,

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^\infty \log \det[e - \hat{k}(\lambda)]\, d\lambda \\
&\quad = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\log \det[e - \hat{k}(\lambda)]}{\varepsilon^2 \lambda^2 + 1}\, d\lambda \\
&\quad = -\lim_{\epsilon \downarrow 0} \lim_{R \uparrow \infty} \left\{ \frac{1}{2\pi} \int_{-R}^R \frac{\log \det[e + \hat{b}(\lambda)]}{\varepsilon^2 \lambda^2 + 1}\, d\lambda + \frac{1}{2\pi} \int_{-R}^R \frac{\log \det[e + \hat{a}(\lambda)]}{\varepsilon^2 \lambda^2 + 1}\, d\lambda \right\} \\
&\quad = -\lim_{\epsilon \downarrow 0} \left\{ \frac{\log \det[e + \hat{b}(-i/\varepsilon)]}{2\varepsilon} + \frac{\log \det[e + \hat{a}(i/\varepsilon)]}{2\varepsilon} \right\} \\
&\quad = -\operatorname{trace} \left[\frac{a(0) + b(0)}{2} \right] \\
&\quad = -\operatorname{trace} \Gamma_0^\infty(0, 0).
\end{aligned}$$

See, e.g., the proof of Theorem 7.4 in Dym and Gohberg [8] for additional details on the last part of the argument if need be, but bear in mind that in the present case k belongs to both $L^1_{n\times n}$ and $L^2_{n\times n}$ and is continuous. ∎

For different proofs of the same basic identity (4.2) see Mikaelan [15] and Lemma 4.1 of Dym [7]. The link with the latter is provided by Theorem 3.2 and the corollary to Lemma 4.1 of the present paper. For scalar variants see Devinatz [3] and Hirschman [12].

THEOREM 4.1. *If K is the integral operator on $L^2_n[0, \infty)$ with kernel $K(t, s) = k(t-s)$ and if k and $\hat{k}$ belong to $L^1_{n\times n}(\mathbb{R})$ and*

$$\int_{-\infty}^{\infty} |s| \operatorname{trace}\{k(s)[k(s)]^*\}\, ds < \infty \tag{4.3}$$

and if $I-K$ and $I - P_T K P_T$ are invertible on $L^2_n[0, \infty)$ for every $T \geqslant 0$, then $P_T K P_T$ is trace class for every $T \geqslant 0$ and

$$\begin{aligned} \alpha_T &= \frac{\det(I - P_T K P_T)}{\exp\left\{\dfrac{T}{2\pi}\displaystyle\int_{-\infty}^{\infty} \log\det[e - \hat{k}(\lambda)]\, d\lambda\right\}} \\ &= \exp\left\{\operatorname{trace}\int_0^T \left[\int_t^\infty \Gamma_0^s(0, s)\,\Gamma_0^s(s, 0)\, ds\right] dt\right\}. \end{aligned} \tag{4.4}$$

Moreover, the limit as $T \uparrow \infty$ exists and is finite. If

$$\varepsilon(T) = T[\Gamma_0^\infty(0, 0) - \Gamma_0^T(0, 0)] \tag{4.5}$$

tends to zero as $T \uparrow \infty$, then

$$\lim_{T\uparrow\infty} \alpha_T = \exp\left\{\int_0^\infty \operatorname{trace}[s\,\Gamma_0^s(0, s)\,\Gamma_0^s(s, 0)\, ds]\right\}. \tag{4.6}$$

Proof. For a proof that $P_T K P_T$ is trace class and that $Q_T K P_T$ and $P_T K Q_T$ are Hilbert–Schmidt, see Lemmas 3.1 (with $G = \hat{k}$ and $J_\gamma^T(\gamma, \gamma)\, d\Delta(\gamma) = d\gamma$) and 2.3 of Dym [7], respectively. The invertibility of $I-K$ on $L^2_n[0, \infty)$, which is equivalent to the invertibility of $I - Q_t K Q_t$ for translation kernels, guarantees that $I-K$ admits an invertible left factorization of the form (2.1). This is perhaps seen most easily by invoking the spectral factorization (4.2) and identifying the transforms of the factors in (2.1) via the formulas

$$[(I + X_-)f]^\wedge = (e + \hat{a})^{-1}\hat{f}, \tag{4.7}$$

$$[(I + Y_-)f]^\wedge = (e + \hat{a})\hat{f}, \tag{4.8}$$

$$[(I+X_+)f]^\wedge = \mathfrak{p}(e+\hat{b})^{-1}\hat{f}, \tag{4.9}$$

$$[(I+Y_+)f]^\wedge = \mathfrak{p}(e+\hat{b})\hat{f}, \tag{4.10}$$

in which $f \in L_n^2[0, \infty)$ and $\mathfrak{p}$ denotes the orthogonal projection of $L_n^2(\mathbb{R})$ onto the Hardy space $H_n^2 = \{L_n^2[0, \infty)\}^\wedge$.

Therefore Theorems 3.1 and 3.2 and formula (3.1) are applicable. In particular, by Theorem 3.2, Corollary 1 to Lemma 4.1 and Lemma 4.2,

$$\det[(I+P_TX_+P_T)(I+P_TX_-P_T)] = \exp\left\{\frac{T}{2\pi}\int_{-\infty}^{\infty} \log\det[e-\hat{k}(\lambda)]\,d\lambda\right\},$$

whereas, by Lemma 4.1,

$$\begin{aligned}
&\operatorname{trace}\int_0^T\left[\int_T^\infty \Gamma_t^s(t,s)\,\Gamma_t^s(s,t)\,ds\right]dt\\
&= \operatorname{trace}\int_0^T\left[\int_T^\infty \Gamma_0^{s-t}(0,s-t)\,\Gamma_0^{s-t}(s-t,0)\,ds\right]dt\\
&= \operatorname{trace}\int_0^T\left[\int_{T-t}^\infty \Gamma_0^\tau(0,\tau)\,\Gamma_0^\tau(\tau,0)\,d\tau\right]dt\\
&= \operatorname{trace}\int_0^T\left[\int_t^\infty \Gamma_0^\tau(0,\tau)\,\Gamma_0^\tau(\tau,0)\,d\tau\right]dt.
\end{aligned}$$

Formula (4.4) is now immediate from (3.1). The existence of a finite limit as $T \uparrow \infty$ is guaranteed by Theorem 4.3 below which is in force under the present assumptions.

Finally, the identity

$$\begin{aligned}
&\int_0^T\left[\int_t^\infty \Gamma_0^s(0,s)\,\Gamma_0^s(s,0)\,ds\right]dt\\
&= \int_0^T\left[\int_t^T \Gamma_0^s(0,s)\,\Gamma_0^s(s,0)\,ds\right]dt + T\left[\int_T^\infty \Gamma_0^s(0,s)\,\Gamma_0^s(s,0)\,ds\right]\\
&= \int_0^T s\,\Gamma_0^s(0,s)\,\Gamma_0^s(s,0)\,ds + \varepsilon(T)
\end{aligned}$$

serves to justify the final statement. ∎

We remark that $\varepsilon(T)$, as defined in (4.5), tends to zero as $T \uparrow \infty$ if the operators $(I-P_TKP_T)^{-1}$ restricted to $P_TL_n^2[0, \infty)$ are bounded uniformly for $T \geqslant T_0$ for some $T_0 > 0$. This will certainly be the case if $\|k\|_1 < \frac{1}{2}$ as was assumed by Achiezer [1]. In fact a necessary and sufficient condition for this is that $e-\hat{k}$ admit both a left and a right spectral factorization; see

Theorems 6.1 and 6.2 on pages 206, 207 of Gohberg and Feldman [11]; Lemma 2.4 of Dym [6] may also be helpful. The implications of these remarks are summarized in the next theorem.

THEOREM 4.2. *If K is an integral operator on $L_n^2[0, \infty)$ with kernel $K(t, s) = k(t-s)$ and if k and $\hat{k}$ belong to $L^1_{n \times n}(\mathbb{R})$ and (4.3) holds, if $e - \hat{k}$ admits both a left and a right spectral factorization and if $I - P_T K P_T$ is invertible for every $T \geqslant 0$, then*

$$\lim_{T \uparrow \infty} a_T = \exp \left\{ \int_0^\infty \operatorname{trace} s\, \Gamma_0^s(0, s)\, \Gamma_0^s(s, 0, ds) \right\} . \tag{4.11}$$

Proof. Theorem 4.1 is applicable because $I - K$ is invertible on $L_n^2[0, \infty)$ if $e - \hat{k}$ admits a right spectral factorization. It thus suffices to show that $\varepsilon(T)$, as defined in (4.5), tends to zero as $T \uparrow \infty$. But, in terms of the notation introduced in Dym and Gohberg [8],

$$\begin{aligned} \varepsilon(T) &= T[a(0+) - a_T(0+)] \\ &= \frac{T}{2\pi} \int_{-\infty}^{\infty} [\hat{b}(\lambda) - \hat{b}_T(\lambda)][e - \hat{k}(\lambda)][\hat{a}(\lambda) - \hat{a}_T(\lambda)]\, d\lambda; \end{aligned}$$

the last formula is (6.10) of (8) with $\tau = \infty$ and $\mu = T$; see also pages 211, 212, where the derivation for the special case in which $b(\lambda) = [a(\lambda)]^\times$ is discussed. Therefore,

$$\begin{aligned} |\varepsilon(T)| &\leqslant \text{const. } T \|\hat{b} - \hat{b}_T\|_2 \|\hat{a} - \hat{a}_T\|_2 \\ &\leqslant \text{const. } T[\|\hat{b} - \hat{b}_T\|_2^2 + \|\hat{a} - \hat{a}_T\|_2^2]. \end{aligned}$$

The next step is to observe that

$$\begin{aligned} \|\hat{a} - \hat{a}_T\|_2^2 &\leqslant \text{const. } \|a - a_T\|_2^2 \\ &= \text{const. } \left\{ \|P_T(a - a_T)\|_2^2 + \int_T^\infty |a(s)|^2\, ds \right\} \\ &\leqslant \text{const. } \int_T^\infty |a(s)|^2\, ds, \end{aligned}$$

since

$$P_T(a - a_T) = (I - P_T K P_T)^{-1} P_T \varphi,$$

where

$$\varphi(t) = \int_T^\infty k(t-s)\, a(s)\, ds,$$

and

$$\|P_T(a - a_T)\|_2^2 \leqslant \text{const. } \|P_T\varphi\|_2^2$$

$$\leqslant \text{const.} \int_T^\infty |a(s)|^2\, ds,$$

thanks to the uniform boundedness of $(I - P_TKP_T)^{-1}$ on $P_TL_n^2[0, \infty)$ for large T under the given assumptions (see the discussion preceding the theorem) and some elementary estimates. Combining bounds it now follows readily that

$$T\|\hat{a} - \hat{a}_T\|_2^2 \leqslant \text{const. } T\int_T^\infty |a(s)|^2\, ds$$

$$\leqslant \text{const.} \int_T^\infty s\,|a(s)|^2\, ds = o(1),$$

as $T \uparrow \infty$, whereas, by a similar analysis,

$$T\|\hat{b} - \hat{b}_T\|_2^2 \leqslant \text{const.} \int_{-\infty}^T |s|\,|b(s)|^2\, ds = o(1)$$

as $T \uparrow \infty$. This finishes the proof though, for the sake of completeness, we recall that

$$\int_0^\infty s\,|a(s)|^2\, ds + \int_{-\infty}^0 |s|\,|b(s)|^2\, ds < \infty$$

because $\mathfrak{p}\hat{a}\mathfrak{q}$ and $\mathfrak{q}\hat{b}\mathfrak{p}$ ($\mathfrak{q} = I - \mathfrak{p}$) are Hilbert–Schmidt on $L_n^2(\mathbb{R})$ as follows from (4.2) and the fact that $\mathfrak{p}\hat{b} = \mathfrak{p}\hat{b}\mathfrak{p}$ and $\hat{a}\mathfrak{p} = \mathfrak{p}\hat{a}\mathfrak{p}$; see the proof of Lemma 2.2 for the argument. ∎

We remark that (4.11) expresses the limit in the form found by Achiezer [1] in the scalar case and Mikaelan [15] in the matrix case. If, moreover, $k(-t) = [k(t)]^\times$, as was assumed by Achiezer and Mikaelan, then

$$\Gamma_0^s(0, s) = [\Gamma_0^s(s, 0)]^\times$$

and the integral in (4.11) converges absolutely. The estimates in the proof are adapted from Devinatz's treatment [3] of the scalar case.

We further remark that if Z, as defined in Theorem 2.1, is of trace class, then by Theorem 2.1,

$$\alpha_\infty = \lim_{T \uparrow \infty} \alpha_T = \det[(I + Y_-)(I + X_+)(I + X_-)(I + Y_+)].$$

But, by (4.7)–(4.10),

$$\begin{aligned}[(I+Y_-)(I+X_+)(I+X_-)(I+Y_+)f]^\wedge &= (e+\hat{a})\,\mathfrak{p}(e+\hat{b})^{-1}(e+\hat{a})^{-1}\,\mathfrak{p}(e+\hat{b})\,\hat{f}\\ &= \mathfrak{p}(e+\hat{a})\,\mathfrak{p}(e+\hat{b})^{-1}\mathfrak{p}(e+\hat{a})^{-1}\,\mathfrak{p}(e+\hat{b})\,\mathfrak{p}\hat{f}\end{aligned}$$

for every $f \in L_n^2[0,\infty)$ and so the determinant of interest is equal to

$$\begin{aligned}&\det[\mathfrak{p}(e+\hat{a})\,\mathfrak{p}(e+\hat{b})^{-1}\mathfrak{p}(e+\hat{a})^{-1}\mathfrak{p}(e+\hat{b})\,\mathfrak{p}]\\ &\quad= \det[\mathfrak{p}(e+\hat{b})^{-1}\mathfrak{p}(e+\hat{a})^{-1}\mathfrak{p}(e+\hat{b})\,\mathfrak{p}(e+\hat{a})\,\mathfrak{p}]\\ &\quad= \det[\mathfrak{p}(e-\hat{k})\,\mathfrak{p}(e+\hat{b})(e+\hat{a})\,\mathfrak{p}].\end{aligned}$$

If $n=1$, then the last term is equal to

$$\det[\mathfrak{p}(e-\hat{k})\,\mathfrak{p}(e+\hat{a})(e+\hat{b})\,\mathfrak{p}] = \det[\mathfrak{p}(e-\hat{k})\,\mathfrak{p}(e-\hat{k})^{-1}\mathfrak{p}].$$

This last formula for α_∞ is valid more generally. The situation is summarized in the next theorem.

THEOREM 4.3. *If K is an integral operator on $L_n^2[0,\infty)$ such that $K(t,s)=k(t-s)$, where k and $\hat{k}$ belong to $L^1_{n\times n}(\mathbb{R})$ and (4.3) holds and if*

$$\det[e-\hat{k}(\lambda)] \neq 0$$

for every $\lambda \in \mathbb{R}$ and

$$\operatorname{index}\det[e-\hat{k}(\lambda)] = 0,$$

then $P_T K P_T$ is of trace class for every $T \geqslant 0$ and

$$\lim_{T\uparrow\infty} \alpha_T = \det[\mathfrak{p}(e-\hat{k})\,\mathfrak{p}(e-\hat{k})^{-1}\mathfrak{p}]. \tag{4.12}$$

If $n=1$, then

$$\det[\mathfrak{p}(e-\hat{k})\mathfrak{p}(e-\hat{k})^{-1}\mathfrak{p}] = \exp\left[\int_0^\infty x[\log(e-\hat{k})]^\vee(x)[\log(e-\hat{k})]^\vee(-x)\,dx\right]. \tag{4.13}$$

Proof. See Theorem 4.1 and Lemma 4.1 of Dym [7]. ∎

We remark that formula (4.13) was first obtained by Kac [14] under somewhat restrictive conditions by probabilistic arguments. A direct connection between (4.13) and Achiezer's formula was first found by Hirschman [12].

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